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Research Article

Infinitely Many Solutions for a Semilinear Elliptic Equation with Sign-Changing Potential

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We consider a semilinear elliptic equation with sign-changing potential $-\Delta u - V(x)u = f(x, u)$, $u \in H^1(\mathbb{R}^N)$, where $V(x)$ is a function possibly changing sign in \mathbb{R}^N . Under certain assumptions on f , we prove that the equation has infinitely many solutions.

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1. Introduction

In this paper, the existence of solutions of the following elliptic equation:

$$-\Delta u - V(x)u = f(x, u), \quad u \in H^1(\mathbb{R}^N) \quad (P)$$

is studied, where $V(x)$ is a function possibly changing sign, f is a continuous function on $\mathbb{R}^N \times \mathbb{R}$.

Problem (P) arises in various branches of applied mathematics and has been studied extensively in recent years. For example, Rabinowitz [1] has studied the existence of a nontrivial solution of this kind of equation on a bounded domain. Lien et al. [2] studied the existence of positive solutions of problem (P) with $V(x) \equiv \lambda$ (λ is a positive constant) and $f(x, u) = |u|^{p-2}u$. And Grossi et al. [3] established some existence results for $-\Delta u = \lambda u + a(x)g(u)$, where $a(x)$ is a function possibly changing sign, $g(u)$ has superlinear growth and λ is a positive real parameter; he discussed both the cases of subcritical and critical growth for $g(u)$ and proved the existence of linking type solutions.

Cerami et al. [4] prove that the problem (P) has infinitely many solutions, where $a(x)$ is a regular function such that $\liminf_{|x| \rightarrow \infty} a(x) = a_\infty > 0$ and some suitable decay assumptions, $f(x, u) = |u|^{p-2}u$. Kryszewski and Szulkin [5] considered the existence of

a nontrivial solution of (P) in a situation where $f(x, u)$ and $V(x)$ are periodic in the x -variable, $f(x, u)$ is superlinear at $u = 0$ and $\pm\infty$, and 0 lies in a spectral gap of $-\Delta u + V$. If in addition $f(x, u)$ is odd in u , (P) has infinitely many solutions.

In [6], Zeng and Li proved existence of $m - n$ pairs of nontrivial solutions ($m > n$, m and n are integers) of (P) , under the assumption that $V(x)$ is a function possibly changing sign in \mathbb{R}^N and $f(x, u)$ satisfies some growth conditions.

In this paper, we prove the existence of infinitely many solutions of (P) , under the assumption that $V(x)$ is a function possibly changing sign in \mathbb{R}^N and $f(x, u)$ also satisfies some growth conditions. One difficulty in considering problem (P) is the loss of compactness because of \mathbb{R}^N ; the other is that $V(x)$ may change sign, which leads to difficulty in verifying the Palais-Smale condition and applying the well-known theorem.

Notation. We use the following notations. A strip region is a domain like this: for $d > 0$, $\tilde{\Omega} = \{x \in \mathbb{R}^N; -d < x_i < d \text{ at least for some fixed } i\}$. $V(x) = V^+(x) - V^-(x)$, where $V^\pm = \max\{\pm V(x), 0\}$. $\Omega_1 = \{x \in \mathbb{R}^N; V^-(x) \neq 0\}$, $\Omega_2 = \{x \in \mathbb{R}^N; V^-(x) = 0\}$.

X is defined as the completion of $D(\mathbb{R}^N)$ with respect to the inner product

$$\langle u, v \rangle_1 := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V^-(x)uv) dx. \quad (1.1)$$

The functional associated with (P) is

$$I(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V^-(x)u^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} V^+(x)u^2 dx - \int_{\mathbb{R}^N} F(x, u) dx, \quad (1.2)$$

for $u \in X$, where $F(x, u) = \int_0^u f(x, t) dt$.

Our fundamental assumptions are as follows:

- (A₁) $V^+(x) \in L^{N/2}(\mathbb{R}^N)$, $\text{meas}\{x \in \mathbb{R}^N; V^+(x) \neq 0\} > 0$. $V^-(x) \in L^\infty(\mathbb{R}^N)$, Ω_2 is a strip region, $\lim_{|x| \rightarrow \infty} V^-(x) = a > 0$ in Ω_1 .
- (A₂) $f \in C(\mathbb{R}^N \times \mathbb{R})$ and there are constants $C_1 > 0$ and $2 < p \leq q < 2^*$ such that $|f(x, t)| \leq C_1(|t|^{p-1} + |t|^{q-1})$.
- (A₃) There exists $\alpha > 2$ such that $0 < \alpha F(x, t) \leq t f(x, t)$ for every $x \in \mathbb{R}^N$ and $t \neq 0$.
- (A₄) $\lim_{|x| \rightarrow \infty} \sup_{|t| \leq r} (|f(x, t)|/|t|) = 0$ for every $r > 0$.
- (A₅) For any $t \in \mathbb{R}$, $f(x, t) = -f(x, -t)$.

Here 2^* denotes the critical Sobolev exponent, that is, $2^* = 2N/(N - 2)$ for $N \geq 3$ and $2^* = \infty$ for $N = 1, 2$.

Theorem 1.1. *Under the assumptions (A₁)–(A₅), (P) possesses infinitely many solutions on X .*

Remark 1.2. It is easily seen that (A₂)–(A₅) hold for nonlinearities of the form $f(x, t) = \sum_{i=1}^k a_i(x)|t|^{p_i-2}t$ with $2 < p_i < 2^*$ and for $i = 1, \dots, k$, the nonnegative function $a_i(x) \in L^\infty(\mathbb{R}^N)$, $\lim_{|x| \rightarrow \infty} a_i(x) = 0$.

2. Preliminaries

We define the Palais-Smale (denoted by (PS)) sequences, (PS) -values, and (PS) -conditions in X for I as follows.

Definition 2.1 (cf. [7]). (i) For $c \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(PS)_c$ -sequence in X for I if $I(u_n) = c + o(1)$ and $I'(u_n) = o(1)$ strongly in X' as $n \rightarrow \infty$;

(ii) $c \in \mathbb{R}$ is a (PS) -value in X for I if there is a $(PS)_c$ -sequence in X for I ;

(iii) I satisfies the $(PS)_c$ -condition in X if every $(PS)_c$ -sequence in X for I contains a convergent subsequence;

(iv) I satisfies the (PS) -condition in X if for every $c \in \mathbb{R}$, I satisfies the $(PS)_c$ -condition in X .

Lemma 2.2 (cf. [6, Lemma 2.1]). *Under the assumption (A_1) , the inner product*

$$\langle u, v \rangle_1 := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V^-(x)uv) dx \quad (2.1)$$

is well defined; therefore the corresponding norm $\|u\|_1 := \sqrt{\langle u, u \rangle_1}$ is well defined too, which is equivalent to the norm $\|u\| = (\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2 dx))^{1/2}$.

Lemma 2.3 (cf. [8]). *Under the assumption that $V^+(x) \in L^{N/2}(\mathbb{R}^N)$ for the eigenvalue problem*

$$-\Delta u + V^-(x)u = \mu V^+(x)u, \quad u \in E \quad (2.2)$$

there exists a sequence of eigenvalues $\mu_n \rightarrow \infty$ such that the eigenfunction sequence φ_n is an orthonormal basis of E .

When $(PS)_c$ -condition is satisfied for all $c \in \mathbb{R}$, there are known methods of obtaining an unbounded sequence of critical values of φ (see, e.g., [9]).

Theorem 2.4 (cf. [10, Theorem 6.5]). *Suppose that E is an infinite-dimensional Banach space and suppose $\varphi \in C^1(E, \mathbb{R})$ satisfies (PS) -condition, $\varphi(u) = \varphi(-u)$ for all u , and $\varphi(0) = 0$. Suppose $E = E^- \oplus E^+$, where E^- is finite dimensional, and assume the following conditions:*

(i) *there exist $\zeta > 0$ and $Q > 0$ such that if $\|u\| = Q$ and $u \in E^+$, then $\varphi(u) \geq \zeta$;*

(ii) *for any finite-dimensional subspace $W \subset E$ there exists $R = R(W)$ such that $\varphi(u) \leq 0$ for $u \in W$, $\|u\| \geq R$.*

Then φ possesses an unbounded sequence of critical values.

3. The $(PS)_c$ -Condition

Lemma 3.1. *Under the assumptions (A_1) , (A_2) , and (A_3) , for every $c \in \mathbb{R}$, any $(PS)_c$ -sequence is bounded.*

Proof. By the eigenvalue problem in Lemma 2.3, there exist $k \in \mathbb{N}$ such that eigenvalues are $\mu_1 < \mu_2 \leq \mu_3 \leq \dots \leq \mu_k \leq \lambda < \mu_{k+1} \leq \dots$ for some $\lambda \geq 1$; the corresponding eigenfunction

is $\varphi_1, \varphi_2, \varphi_3, \dots, \varphi_k, \varphi_{k+1}, \dots$, then we denote $X = X_1 \oplus X_2$, with $X_1 = \bigoplus_{i=1}^k \text{span}\{\varphi_i\}$, $X_2 = X_1^\perp$, and denote $u_n \in X$ as $u_n = v_n + w_n$, where $v_n \in X_1$, $w_n \in X_2$. It's obvious that

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + V^-(x)u^2 - \lambda V^+(x)u^2) dx \leq 0, \quad \forall u \in X_1, \quad (3.1)$$

and there exist $\delta > 0$ such that

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + V^-(x)u^2 - V^+(x)u^2) dx \geq \delta \|u\|_1^2, \quad \forall u \in X_2 \quad (3.2)$$

by Lemma 2.3. For any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that $|F(x, u)| \geq C_\epsilon |u|^\alpha - \epsilon |u|^2$ from (A_2) and (A_3) . Choose $2 < \alpha' < \alpha$, then

$$\begin{aligned} & \int_{\mathbb{R}^N} F(x, u_n) dx - \frac{1}{\alpha'} \int_{\mathbb{R}^N} u_n f(x, u_n) dx \\ & \leq \int_{\mathbb{R}^N} \left(1 - \frac{\alpha}{\alpha'}\right) F(x, u_n) dx \\ & \leq \left(1 - \frac{\alpha}{\alpha'}\right) \int_{\mathbb{R}^N} (C_\epsilon |u_n|^\alpha - \epsilon |u_n|^2) dx. \end{aligned} \quad (3.3)$$

Let $\{u_n\}$ be the sequence such that $I(u_n) \rightarrow c$, $I'(u_n) \rightarrow 0$. By inequality (3.2) and $u_n = v_n + w_n$, $v_n \in X_1$, $w_n \in X_2$, and then

$$\begin{aligned} c + 1 + \|u\|_1 & \geq I(u_n) - \frac{1}{\alpha'} \langle I'(u_n), u_n \rangle \\ & = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_n|^2 - V(x)u_n^2) dx - \int_{\mathbb{R}^N} F(x, u_n) dx \\ & \quad - \frac{1}{\alpha'} \int_{\mathbb{R}^N} (|\nabla u_n|^2 - V(x)u_n^2) dx + \frac{1}{\alpha'} \int_{\mathbb{R}^N} u_n f(x, u_n) dx \\ & = \left(\frac{1}{2} - \frac{1}{\alpha'}\right) \int_{\mathbb{R}^N} (|\nabla w_n|^2 - V(x)w_n^2 + |\nabla v_n|^2 - V(x)v_n^2) dx \\ & \quad - \int_{\mathbb{R}^N} F(x, u_n) dx + \frac{1}{\alpha'} \int_{\mathbb{R}^N} u_n f(x, u_n) dx \\ & \geq \left(\frac{1}{2} - \frac{1}{\alpha'}\right) \delta \|w_n\|_1^2 + \left(\frac{1}{2} - \frac{1}{\alpha'}\right) \|v_n\|_1^2 - \left(\frac{1}{2} - \frac{1}{\alpha'}\right) \int_{\mathbb{R}^N} (V^+(x)|v_n|^2) dx \\ & \quad + \left(\frac{\alpha}{\alpha'} - 1\right) \int_{\mathbb{R}^N} (C_\epsilon |u_n|^\alpha - \epsilon |u_n|^2) dx. \end{aligned} \quad (3.4)$$

Choose $\epsilon > 0$ small, then for suitable C_2, C_3 , the above inequality becomes

$$c + 1 + \|u\|_1 \geq C_2 \|u_n\|_1^2 + C_3 |u_n|_\alpha^\alpha - \left(\frac{1}{2} - \frac{1}{\alpha'}\right) |V^+|_{N/2} |v_n|_{2^*}^2. \quad (3.5)$$

Due to $\alpha > 2$, it follows that $\{u_n\}$ is bounded. \square

The following lemma is the same as [6, Lemma 3.2]. For the completeness, we prove it.

Lemma 3.2. *Under the assumptions (A_1) , (A_2) , (A_3) , and (A_4) , I satisfies the (PS)-condition in X .*

Proof. By Lemma 3.1, we know that any $(PS)_c$ sequence u_n is bounded in X . Up to a subsequence, we may assume that $u_n \rightharpoonup u$ in X . In order to establish strong convergence it suffices to show

$$\|u_n\|_1 \longrightarrow \|u\|_1. \quad (3.6)$$

Since $\langle I'(u_n), u_n - u \rangle \rightarrow 0$, we infer that

$$\begin{aligned} 0 &\leq \limsup_{n \rightarrow \infty} (\|u_n\|_1^2 - \|u\|_1^2) \\ &= \limsup_{n \rightarrow \infty} (u_n, u_n - u) \\ &= \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(x, u_n)(u_n - u) dx. \end{aligned} \quad (3.7)$$

We restrict our attention to the case $N \geq 3$, but the cases $N = 1, 2$ can be treated similarly. Let $\epsilon > 0$, for $r \geq 1$, then

$$\begin{aligned} \int_{|u_n| \geq r} f(x, u_n)(u_n - u) dx &\leq C_4 \int_{|u_n| \geq r} |u_n|^{p-1} |u_n - u| dx \\ &\leq C_4 r^{p-2^*} \int_{|u_n| \geq r} |u_n|^{2^*-1} |u_n - u| dx \\ &\leq C_4 r^{p-2^*} |u_n|_{2^*}^{2^*-1} \|u_n - u\|_{2^*}. \end{aligned} \quad (3.8)$$

Since $p < 2^*$, we may fix r large enough such that

$$\int_{|u_n| \geq r} f(x, u_n)(u_n - u) dx \leq \frac{\epsilon}{3} \quad (3.9)$$

for all n . Moreover, by (A_4) there exists $R_1 > 0$ such that

$$\int_{(|u_n| \leq r \cap |x| \geq R_1)} f(x, u_n)(u_n - u) dx \leq |u_n|_2 \|u_n - u\|_2 \sup_{|t| \leq r, |x| \geq R_1} \frac{|f(x, t)|}{|t|} \leq \frac{\epsilon}{3} \quad (3.10)$$

for all n . Finally, since $u_n \rightarrow u$ in $L^s(B_{R_1}(0))$ for $s \in [2, 2^*)$, we can use (\mathbb{A}_2) again to derive

$$\int_{(|u_n| \leq r \cap |x| \leq R_1)} f(x, u_n)(u_n - u) dx \leq \frac{\epsilon}{3} \quad (3.11)$$

for n large enough. Combining (3.9)–(3.11) we conclude that

$$\int_{\mathbb{R}^N} f(x, u_n)(u_n - u) dx \leq \epsilon \quad (3.12)$$

for n large enough. From this and (3.7), we deduce (3.6) and complete the proof. \square

4. Infinitely Many Solutions

We can obtain an infinite sequence of critical values from Theorem 2.4.

Proof of Theorem 1.1. We apply Theorem 2.4 with $E = X$, $\varphi = I$. It is clear that $I \in C^1(X, \mathbb{R})$ is even because of (\mathbb{A}_1) , (\mathbb{A}_2) , and (\mathbb{A}_5) . $I(0) = 0$. By lemma 3.2, the (PS) -condition is satisfied. From the proof of Lemma 3.1, we have $X = X_1 \oplus X_2$, where $X_1 = \bigoplus_{i=1}^k \text{span}\{\varphi_i\}$, $X_2 = X_1^\perp$. That is $E^- = X_1$, $E^+ = X_2$. We only need to check conditions (i) and (ii).

Integrating (\mathbb{A}_2) , there is a constant $C_5 > 0$ such that for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$,

$$|F(x, t)| \leq C_5(|t|^p + |t|^q). \quad (4.1)$$

By the Sobolev embedding theorem and (3.2), we have the estimate

$$\begin{aligned} I(u) &\geq \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V^-(x)u^2) dx - \frac{1}{2} \int_{\mathbb{R}^N} V^+(x)u^2 dx - C_5 \int_{\mathbb{R}^N} (|u|^p + |u|^q) dx \\ &\geq \frac{\delta}{2} \|u\|_1^2 - C_6 \|u\|_1^p - C_7 \|u\|_1^q \end{aligned} \quad (4.2)$$

for $u \in X_2$. Let $\|u\|_1 = \varrho$ and $u \in X_2$,

$$I(u) \geq \frac{\delta}{2} \varrho^2 - C_6 \varrho^p - C_7 \varrho^q > 0 \quad (4.3)$$

for small ϱ . Thus condition (i) is fulfilled with $\zeta = (\delta/2)\varrho^2 - C_6\varrho^p - C_7\varrho^q$.

By (\mathbb{A}_3) , there is a constant C_8 such that $|F(x, t)| \geq C_8|t|^\alpha$ for every $x \in \mathbb{R}^N$ and $|t| > \epsilon$. Indeed, let $\epsilon > 0$ small be given. By integration of (\mathbb{A}_3) , we have for $x \in \mathbb{R}^N$ and $|t| > \epsilon$,

$$F(x, t) \geq \frac{F(x, \epsilon)}{\epsilon^\alpha} |t|^\alpha \geq C_8 |t|^\alpha. \quad (4.4)$$

Let W be a finite-dimensional subspace of X . Since all norms are equivalent of W and since

$$I(u) \leq \frac{1}{2}\|u\|_1^2 - \frac{1}{2} \int_{\mathbb{R}^N} V^+ u^2 dx - C_9 \|u\|_\alpha^\alpha. \quad (4.5)$$

Also since $\alpha > 2$, condition (ii) follows. Thus we complete the proof. \square

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